Inequalities I: Tedious Techniques

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1. (IMO '95) Let a, b, c be positive real numbers with $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

2. (USAMO '03) Let a, b, c be positive real numbers. Prove that
$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8.$$

3. (Bulgaria '97) Let a, b, c be positive real numbers such that $abc = 1$. Prove that
$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

4. (USAMO '98) Let a_0, a_1, $\ldots$, a_n be numbers from the interval $(0, \pi/2)$ such that
$$\tan(a_0 - \tfrac{\pi}{4}) + \tan(a_1 - \tfrac{\pi}{4}) + \cdots + \tan(a_n - \tfrac{\pi}{4}) \geq n - 1.$$
Prove that
$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.$$

5. (China '97) Let x_1, x_2, $\ldots$, x_{1997} be real numbers satisfying the following conditions:
   (a) $-\frac{1}{\sqrt{3}} \leq x_i \leq \sqrt{3}$ for $i = 1, 2, \ldots, 1997$;
   (b) $x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}$.

6. Find the maximum number of edges a k-partite graph on n edges can contain.

7. (IMO '99) Let $n \geq 2$ be a fixed integer. Find the smallest constant C such that for all nonnegative reals x_1, $\ldots$, x_n,
$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \biggl(\sum_{1 \leq i \leq n} x_i \biggr)^4.$$
Determine when equality occurs.

8. (Vietnam '96) Let a, b, c, d be four nonnegative real numbers satisfying the condition
$$2(ab+ac+ad+bc+bd+cd) + abc + abd + acd + bcd = 16.$$
Prove that
$$a + b + c + d \geq \frac{2}{3}(ab+ac+ad+bc+bd+cd)$$
and determine when equality holds.